

# A differential in the Lyndon-Hochschild-Serre spectral sequence

by I. J. Leary\*

School of Mathematical Sciences,  
 Queen Mary and Westfield College,  
 Mile End Road,  
 London  
 E1 4NS

## Abstract.

We consider the Lyndon-Hochschild-Serre spectral sequence with  $\mathbf{F}_p$  coefficients for a central extension with kernel cyclic of order a power of  $p$  and arbitrary discrete quotient group. For this spectral sequence  $d_2$  and  $d_3$  are known, and we give a description for  $d_4$ . Using this result we deduce a similar formula for the Serre spectral sequence for a fibration with fibre  $BC_{p^n}$ . The differential from odd rows to even rows involves a Massey triple product, so we describe the calculation of such products in the cohomology of a finite abelian group. As an example we determine the Poincaré series for the mod-3 cohomology of various 3-groups.

## Introduction.

Let  $G$  be a discrete group, let  $C$  be a cyclic group of order  $p^m$  for some prime  $p$  and  $m \geq 1$ , and let  $E$  be a central extension with kernel  $C$  and quotient  $G$ . Such an extension is described by its extension class, which may be any element of  $H^2(G; C)$ . Since  $C$  is central, the  $E_2$ -page of the Lyndon-Hochschild-Serre spectral sequence with  $\mathbf{F}_p$  coefficients for this extension is isomorphic to the graded tensor product of  $H^*(G)$  and  $H^*(C)$ . (Note that throughout this paper  $H^*(.)$  will denote mod- $p$  cohomology.) There is an isomorphism

$$H^*(C) \cong \mathbf{F}_p[u, t]/I,$$

where  $u$  has degree one,  $t$  has degree two, and  $I$  is generated by  $u^2$  except when  $p^m = 2$  in which case  $I$  is generated by  $u^2 - t$ . In particular this implies that each row of the  $E_2$ -page of the spectral sequence is isomorphic to a single copy of  $H^*(G)$ .

The first two differentials in the spectral sequence have been known since the calculations of the cohomology of Eilenberg-MacLane spaces by Cartan and Serre [2], [9]. Modulo choice of various isomorphisms,  $d_2(u)$  is the mod- $p$  reduction of the extension class,  $d_2(t)$  is zero, and  $d_3(t)$  is the image of the extension class under the Bockstein for the following coefficient sequence.

$$0 \rightarrow \mathbf{F}_p \rightarrow \mathbf{Z}/p^{m+1} \rightarrow \mathbf{Z}/p^m \cong C \rightarrow 0$$

We shall describe the next differential  $d_4$ , in Theorem 3, but first we require two lemmata concerning the spectral sequence. In Corollary 4 we deduce a similar result for the Serre spectral sequence for a fibration with fibre  $BC_{p^m}$ .

---

\* supported by SERC post-doctoral fellowship number B90 RFH 8960

### The spectral sequence.

We set up the Lyndon-Hochschild-Serre spectral sequence as described in Evens' book [5]. Thus we let  $P_*$  be the standard (or bar) resolution for  $\mathbf{F}_p$  over  $\mathbf{F}_p G$ , and  $Q_*$  be the standard resolution for  $\mathbf{F}_p$  over  $\mathbf{F}_p E$ . Now we define

$$E_0^{i,j} = \text{Hom}_E(P_i \otimes Q_j, \mathbf{F}_p),$$

where  $E$  acts diagonally on  $P_* \otimes Q_*$ ,  $d_1$  is the map induced by the differential on  $P_*$ , and  $d_0$  is the map induced by the differential on  $Q_*$  except that a sign must be introduced (in fact on  $E_0^{i,j}$   $d_0$  is the adjoint of  $(-1)^i 1 \otimes d^Q$ ). Using the Kunneth theorem it may be checked that the singly graded total complex  $(P_* \otimes Q_*)_*$  with differential  $d_0 + d_1$  is an  $E$ -free resolution for  $\mathbf{F}_p$ .

To introduce products into the spectral sequence we need a diagonal approximation for  $(P_* \otimes Q_*)_*$ , that is a chain map from  $(P_* \otimes Q_*)_*$  to  $(P_* \otimes Q_* \otimes P_* \otimes Q_*)_*$  inducing the identity map on homology. For any two chain complexes  $A_*$  and  $B_*$ , let  $\tau$  denote the chain isomorphism from  $A_* \otimes B_*$  to  $B_* \otimes A_*$  given by  $\tau(a \otimes b) = (-1)^{|a||b|}$ . If  $\Delta_0^P$  and  $\Delta_0^Q$  are diagonal approximations for  $P_*$  and  $Q_*$  respectively, then  $(1 \otimes \tau \otimes 1)(\Delta_0^P \otimes \Delta_0^Q)$  is a diagonal approximation for  $P_* \otimes Q_*$ , which is coassociative provided that  $\Delta_0^P$  and  $\Delta_0^Q$  are. Since it is possible to choose  $\Delta_0^P$  and  $\Delta_0^Q$  to be coassociative using the Alexander-Whitney formula (given on the usual  $\mathbf{F}_p$  basis of  $P_n$  by

$$\Delta_0^P(g_0, \dots, g_n) = \sum_{i=0}^n (g_0, \dots, g_i) \otimes (g_i, \dots, g_n)$$

and similarly for  $\Delta_0^Q$ ) we do so. We thus obtain an associative product (which we shall call the cup product) on the  $E_0$ -page of the spectral sequence given by

$$\phi \smile \theta(x) = \mu(\phi \otimes \theta)(1 \otimes \tau \otimes 1)(\Delta_0^P \otimes \Delta_0^Q)(x)$$

for all  $\phi$  and  $\theta$  in  $E_0$  and all  $x \in P_* \otimes Q_*$ , where  $\mu$  is the product map from  $\mathbf{F}_p \otimes \mathbf{F}_p$  to  $\mathbf{F}_p$ . The only reason why we insisted that  $P_*$  and  $Q_*$  should be the standard resolution was to ensure that  $E_0$  could be endowed with an associative product. The differentials  $d_0$  and  $d_1$  are derivations for this product.

We also need to introduce analogues within the spectral sequence of the cup-1 product [10]. The map  $\tau \Delta_0^P$  is also a diagonal approximation for  $P_*$ , so is chain homotopic to  $\Delta_0^P$ . Let  $\Delta_1^P$  be a chain homotopy from  $\Delta_0^P$  to  $\tau \Delta_0^P$ . For example,  $\Delta_1^P$  could be the map given by Steenrod [10]

$$\Delta_1^P(g_0, \dots, g_n) = \sum_{0 \leq i < j \leq n} (-1)^{f(i,j,n)} (g_0, \dots, g_i, g_j, \dots, g_n) \otimes (g_i, \dots, g_j),$$

where  $f(i, j, n) = n + (n - i - 1)(j - i - 1)$ . We define  $\Delta_1^Q$  similarly. We now consider three different diagonal approximations for  $P_* \otimes Q_*$ , which are defined as follows.

$$(1 \otimes \tau \otimes 1)(\Delta_0^P \otimes \Delta_0^Q) \quad (1 \otimes \tau \otimes 1)(\tau \otimes 1 \otimes 1)(\Delta_0^P \otimes \Delta_0^Q) \quad (1 \otimes \tau \otimes 1)(\tau \otimes \tau)(\Delta_0^P \otimes \Delta_0^Q)$$

As above, each of these induces a product on the  $E_0$ -page of the spectral sequence. By definition the first induces the cup product, and it is easily checked that the third induces the product

$$(\phi, \theta) \mapsto (-1)^{|\phi||\theta|} \theta \smile \phi,$$

where  $|.|$  refers to total degree. The second induces another product which we shall refer to as

$$(\phi, \theta) \mapsto \phi \wedge \theta.$$

It is easily checked that  $(1 \otimes \tau \otimes 1)(\Delta_1^P \otimes \Delta_0^Q)$  is a chain homotopy between the first two diagonal approximations, and that the map defined to be  $(-1)^i(1 \otimes \tau \otimes 1)(\tau \otimes 1 \otimes 1)(\Delta_0^P \otimes \Delta_1^Q)$  on  $P_i \otimes Q_j$  is a chain homotopy between the second two. We may use these chain homotopies to define cup-1 products of bidegrees  $(-1, 0)$  and  $(0, -1)$  respectively, whose properties are summarised in the following lemma.

**Lemma 1.** *For  $\phi, \theta \in E_0$ , and  $x \in P_i \otimes Q_j$ , define products of  $\phi$  and  $\theta$  as below.*

$$\begin{aligned}\phi \smile_{1,0} \theta(x) &= \mu(\phi \otimes \theta)(1 \otimes \tau \otimes 1)(\Delta_1^P \otimes \Delta_0^Q)(x) \\ \phi \smile_{0,1} \theta(x) &= (-1)^i \mu(\phi \otimes \theta)(1 \otimes \tau \otimes 1)(\tau \otimes 1 \otimes 1)(\Delta_0^P \otimes \Delta_1^Q)(x)\end{aligned}$$

Then  $\smile_{1,0}$  and  $\smile_{0,1}$  satisfy the following coboundary formulae.

$$\begin{aligned}d_0(\phi \smile_{1,0} \theta) &= -(d_0 \phi) \smile_{1,0} \theta - (-1)^{|\phi|} \phi \smile_{1,0} (d_0 \theta) \\ d_1(\phi \smile_{1,0} \theta) &= -(d_1 \phi) \smile_{1,0} \theta - (-1)^{|\phi|} \phi \smile_{1,0} (d_1 \theta) + \phi \smile \theta - \phi \wedge \theta \\ d_0(\phi \smile_{0,1} \theta) &= -(d_0 \phi) \smile_{0,1} \theta - (-1)^{|\phi|} \phi \smile_{0,1} (d_0 \theta) + \phi \wedge \theta - (-1)^{|\phi||\theta|} \theta \smile \phi \\ d_1(\phi \smile_{0,1} \theta) &= -(d_1 \phi) \smile_{0,1} \theta - (-1)^{|\phi|} \phi \smile_{0,1} (d_1 \theta).\end{aligned}$$

**Proof.** Left as an exercise for the reader. *QED*

## Differentials.

In any spectral sequence, if  $\phi \in E_0^{i,j}$  is to survive until the  $E_2$ -page,  $d_0(\phi)$  must be zero, and also  $d_1(\phi)$  must represent zero on the  $E_1$ -page, or equivalently there must be  $\phi_1 \in E_0^{i+1,j-1}$  such that  $d_0(\phi_1) = d_1(\phi)$ . In this case  $d_2(\phi)$  is represented by  $d_1(\phi_1)$ . Similarly,  $\phi$  can only survive until the  $E_3$ -page if there is  $\phi_2 \in E_0^{i+2,j-2}$  such that  $d_1(\phi_1) = d_0(\phi_2)$ , in which case  $d_3(\phi)$  is represented by  $d_1(\phi_2)$ . In this way the higher differentials in the spectral sequence may be computed.

Returning to the spectral sequence we are studying, let  $u$  and  $t$  be cochains in  $E_0^{0,1}$  and  $E_0^{0,2}$  respectively, representing the elements of  $E_2^{0,*}$  with the same names (as used in the introduction). The known description of  $d_2$  implies that  $d_0(u) = 0$ , and there is an element  $\theta$  in  $E_0^{1,0}$  such that  $d_1(u) = d_0(\theta)$ , and  $d_1(\theta) = \xi$  is a cocycle representing  $d_2(u)$ . Similarly, there are cochains  $\eta_1 \in E_0^{1,1}$  and  $\eta_2 \in E_0^{2,0}$  such that  $d_0(t) = 0$ ,  $d_1(t) = d_0(\eta_1)$ ,  $d_1(\eta_1) = d_0(\eta_2)$ , and  $d_1(\eta_2) = \xi'$ , where  $\xi'$  is a cocycle representing  $d_3(t)$ . Note that  $\xi$  and  $\xi'$  are universal cycles. The situation may be summarised in the following diagram.

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
u \rightarrow \bullet & & t \rightarrow \bullet \\
\uparrow & & \uparrow \\
\theta \rightarrow \xi & & \eta_1 \rightarrow \bullet \\
& & \uparrow \\
& & \eta_2 \rightarrow \xi'
\end{array}$$

From now on we fix this notation, so that  $u$ ,  $t$ ,  $\theta$ ,  $\xi$ ,  $\eta_1$ ,  $\eta_2$  and  $\xi'$  will always refer to these cochains. Recall from the introduction that the classes represented by  $\xi$  and  $\xi'$  in  $H^*(G)$  may be easily described in terms of the extension class of the group  $E$  in  $H^2(G; C)$ . We are now able to state our second lemma, which describes a sequence of cochains related to  $t^n$ .

**Lemma 2.** *For all  $n \geq 1$  there exist cochains  $\eta_1(n), \dots, \eta_4(n)$ , with  $\eta_i(n) \in E_0^{i,2n-i}$  such that*

$$\begin{aligned}
d_1(t^n) &= d_0(\eta_1(n)) & d_1(\eta_1(n)) &= d_0(\eta_2(n)) \\
d_1(\eta_2(n)) &= nt^{n-1}\xi' + d_0(\eta_3(n)) & d_1(\eta_3(n)) &= n\eta_1(n-1)\xi' + d_0(\eta_4(n)).
\end{aligned}$$

**Remark.** The existence of  $\eta_1(n)$ ,  $\eta_2(n)$  and  $\eta_3(n)$  and the relations between them follow immediately from the fact that  $d_3(t^n)$  is represented by  $nt^{n-1}\xi'$ . Thus the content of this lemma is the existence of  $\eta_4(n)$  and the equation for  $d_1(\eta_3(n))$ .

**Proof.** Define  $\eta_i(0) = \eta_3(1) = \eta_4(1) = 0$ ,  $\eta_1(1) = \eta_1$ , and  $\eta_2(1) = \eta_2$ , where  $\eta_1$  and  $\eta_2$

are as above. Then define  $\eta_i(n)$  inductively by the following formulae.

$$\begin{aligned}
\eta_1(n) &= \eta_1(n-1)t + t^{n-1}\eta_1 \\
\eta_2(n) &= \eta_2(n-1)t + \eta_1(n-1)\eta_1 + t^{n-1}\eta_2 - (n-1)t^{n-2}(\xi' \smile_{1,0} t) \\
\eta_3(n) &= \eta_3(n-1)t + \eta_2(n-1)\eta_1 + \eta_1(n-1)\eta_2 \\
&\quad - (n-1)(\eta_1(n-2)(\xi' \smile_{1,0} t) + t^{n-2}(\xi' \smile_{1,0} \eta_1)) + (n-1)t^{n-2}(\eta' \smile_{0,1} t) \\
\eta_4(n) &= \eta_4(n-1)t + \eta_3(n-1)\eta_1 + \eta_2(n-1)\eta_2 \\
&\quad - (n-1)(\eta_2(n-2)(\xi' \smile_{1,0} t) + \eta_1(n-2)(\xi' \smile_{1,0} \eta_1) + t^{n-2}(\xi' \smile_{1,0} \eta_2)) \\
&\quad + (n-1)(\eta_1(n-2)(\xi' \smile_{0,1} t) + t^{n-2}(\xi' \smile_{0,1} \eta_1))
\end{aligned}$$

The equations given in the statement hold for  $n = 1$ , and may be verified for all  $n$  by induction.  $\text{QED}$

We are now ready to consider  $d_4$ . First we decide which elements of  $E_2$  will survive until  $E_4$ . Let  $\chi$  be an element of  $H^*(G)$ . Since  $d_2(t^n u \chi) = t^n \xi \chi$  we deduce that  $t^n u \chi$  survives until  $E_3$  iff  $\xi \chi = 0$  in  $H^*(G)$ . In this case  $d_3(t^n u \chi) = -nt^{n-1} u \xi' \chi$ , and since  $d_2$  had trivial image in  $E_2^{*,2n-1}$  it follows that  $t^n u \chi$  survives until  $E_4$  iff  $\xi \chi = 0$  in  $H^*(G)$  and either  $p$  divides  $n$  or  $\xi' \chi = 0$  in  $H^*(G)$ . If  $p$  divides  $n$  then since  $u \chi$  is a universal cycle and  $t^p$  survives until  $E_{2p+1}$  (by the Serre transgression theorem) then  $d_4(t^n u \chi) = 0$ . Thus we only need determine  $d_4(t^n u \chi)$  for  $\chi$  such that  $\chi \xi$  and  $\chi \xi'$  are zero. Similarly, for  $t^n \chi$  to survive, we must have that  $d_3(t^n \chi)$  represents zero in  $E_3$ , which will happen if either  $p$  divides  $n$  (in which case  $d_4(t^n \chi) = 0$ ) or if  $\xi' \chi = \xi \chi'$  in  $H^*(G)$  for some  $\chi'$ .

**Theorem 3.** *In the Lyndon-Hochschild-Serre spectral sequence with  $\mathbf{F}_p$  coefficients for a central extension of a cyclic  $p$ -group  $C$  by a group  $G$ , let  $u$  generate  $E_2^{0,1}$ , let  $t$  generate  $E_2^{0,2}$ , and let  $d_2(u) = \xi$ ,  $d_3(t) = \xi'$ .*

a) *If  $\chi \in H^*(G)$  is such that  $\xi \chi = 0$  and  $\xi' \chi = 0$  in  $H^*(G)$ , then for all  $n$ ,  $t^n u \chi$  survives until  $E_4$ , and*

$$d_4(t^n u \chi) = nt^{n-1} \langle \xi', \chi, \xi \rangle,$$

where  $\langle \xi', \chi, \xi \rangle$  is the Massey product (see next section for a description of the Massey product).

b) *If  $\chi \in H^*(G)$  is such that  $\xi' \chi = \xi \chi'$  for some  $\chi'$ , then for all  $n$   $t^n \chi$  survives until  $E_4$ , and*

$$d_4(t^n \chi) = n(n-1)t^{n-2} u \xi' \chi'.$$

**Remark.** The ‘generic’ case of statement b) above for  $C$  of order  $p$  was conjectured by P. H. Kropholler. He made this conjecture by looking for a natural map between the relevant subquotients of  $H^*(G)$ . The author thanks P. H. Kropholler for showing him this conjecture, which inspired the work contained in this paper.

**Proof.** In either case let  $\chi$  be an element of  $E_0^{*,0}$  yielding the element of the same name in  $E_2^{*,0}$ , and in case b) define  $\chi'$  similarly. The proof now splits into two cases. The conditions in case a) are equivalent to the existence of  $\psi$  and  $\psi'$  in  $E_0^{*,0}$  such that  $d_0(\psi) = 0$ ,  $d_0(\psi') = 0$ ,  $d_1(\psi) = \chi \xi$ , and  $d_1(\psi') = \xi' \chi$ . Define cochains  $\phi_0, \dots, \phi_4$  by the

equations

$$\begin{aligned}\phi_0 &= t^n \chi u, \\ \phi_1 &= \eta_1(n) \chi u + t^n (\chi \theta - (-1)^{|\chi|} \psi), \\ \phi_2 &= \eta_2(n) \chi u + \eta_1(n) (\chi \theta - (-1)^{|\chi|} \psi) - n t^{n-1} \psi' u, \\ \phi_3 &= \eta_3(n) \chi u + \eta_2(n) (\chi \theta - (-1)^{|\chi|} \psi) - n \eta_1(n-1) \psi' u - n t^{n-1} \psi' \theta, \\ \phi_4 &= \eta_4(n) \chi u + \eta_3(n) (\chi \theta - (-1)^{|\chi|} \psi) - n \eta_2(n-1) \psi' u - n \eta_1(n-1) \psi' \theta,\end{aligned}$$

where  $\eta_i(j)$  and  $\theta$  are as described during and just before Lemma 2. The following equations may be seen to hold by applying Lemma 2.

$$\begin{aligned}d_0(\phi_0) &= 0, & d_1(\phi_i) &= d_0(\phi_{i-1}) \quad \text{for } i = 0, 1, 2, \\ d_1(\phi_3) &= d_0(\phi_4) - (-1)^{|\chi|} n t^{n-1} (\xi' \psi + \psi' \xi).\end{aligned}$$

Now note that in  $E_2^{*,*}$ ,  $\phi_0$  represents  $(-1)^{|\chi|} t^n u \chi$ , and that  $-(\xi' \psi + \psi' \xi)$  represents the Massey product  $\langle \xi', \chi, \xi \rangle$ .

The proof for case b) is similar. Here the hypotheses give  $\psi \in E_0^{*,0}$  such that  $d_0(\psi) = 0$  and  $d_1(\psi) = \xi' \chi - \xi \chi'$ . Now define cochains  $\phi_0, \dots, \phi_4$  by the following equations.

$$\begin{aligned}\phi_0 &= t^n \chi, \\ \phi_1 &= \eta_1(n) \chi - n t^{n-1} u \chi', \\ \phi_2 &= \eta_2(n) \chi - n \eta_1(n-1) u \chi' - n t^{n-1} (\theta \chi' + \psi), \\ \phi_3 &= \eta_3(n) \chi - n \eta_2(n-1) u \chi' - n \eta_1(n-1) (\theta \chi' + \psi), \\ \phi_4 &= \eta_4(n) \chi - n \eta_3(n-1) u \chi' - n \eta_2(n-1) (\theta \chi' + \psi).\end{aligned}$$

Using Lemma 2 the following equations may be verified.

$$\begin{aligned}d_0(\phi_0) &= 0, & d_1(\phi_i) &= d_0(\phi_{i+1}) \quad \text{for } i = 0, 1, 2, \\ d_1(\phi_3) &= d_0(\phi_4) - n(n-1) t^{n-2} \xi' u \chi'.\end{aligned}$$

*QED*

**Corollary 4.** Let  $C$  be a cyclic  $p$ -group. Then in the Serre spectral sequence with  $\mathbf{F}_p$  coefficients for a fibration with fibre  $BC$  and path connected base space  $d_4$  may be described as in Theorem 3.

**Proof.** For any path connected  $X$ , Kan-Thurston [7] exhibit an aspherical space  $TX$  (in other words a space having  $\pi_i(X) = 0$  unless  $i = 1$ ) and a map from  $TX$  to  $X$  inducing an isomorphism on cohomology for any local coefficients on  $X$ . If we let  $X$  be the base space of the  $BC$  bundle, it follows that the map from  $TX$  to  $X$  induces an isomorphism between the Serre spectral sequence for the original bundle and that for the induced bundle over  $TX$ . It follows from the homotopy long exact sequence that the total space of the induced bundle is also aspherical, and so the Serre spectral sequence for this bundle is isomorphic to the Lyndon-Hochschild-Serre spectral sequence for the extension given by taking fundamental groups. *QED*

**Remark.** The author has not tried to find a direct proof of Corollary 4.

## Massey Products.

We recall first the definition of the Massey (triple) product as in [1]. Let  $P_*$  be any projective  $\mathbf{F}_p G$  resolution for  $\mathbf{F}_p$ , let  $\Delta$  be a diagonal approximation for  $P_*$ , and let  $H$  be a chain homotopy between  $(\Delta \otimes 1)\Delta$  and  $(1 \otimes \Delta)\Delta$ , that is a map from  $P_*$  to  $(P_* \otimes P_* \otimes P_*)^*$  satisfying

$$dH + Hd = (\Delta \otimes 1)\Delta - (1 \otimes \Delta)\Delta.$$

(Of course such a homotopy exists because  $(\Delta \otimes 1)\Delta$  and  $(1 \otimes \Delta)\Delta$  are chain maps between projective resolutions inducing the same map on homology.) If  $\phi_1$ ,  $\phi_2$  and  $\phi_3$  are cochains from  $P_*$  to  $\mathbf{F}_p$ , we may define a triple product  $h(\phi_1, \phi_2, \phi_3)$  by the formula

$$h(\phi_1, \phi_2, \phi_3)(x) = \mu'(\phi_1 \otimes \phi_2 \otimes \phi_3)H(x),$$

where  $\mu'$  is the usual map from  $\mathbf{F}_p \otimes \mathbf{F}_p \otimes \mathbf{F}_p$  to  $\mathbf{F}_p$ . If  $\phi_1$ ,  $\phi_2$  and  $\phi_3$  are cocycles then it is easily checked that  $dh(\phi_1, \phi_2, \phi_3)$  is  $(\phi_1\phi_2)\phi_3 - \phi_1(\phi_2\phi_3)$ . If also  $\phi_1\phi_2$  and  $\phi_2\phi_3$  are coboundaries, with  $d\psi_1 = \phi_1\phi_2$  and  $d\psi_2 = \phi_2\phi_3$ , then it may be verified that the following cochain is a cocycle.

$$\langle \phi_1, \phi_2, \phi_3 \rangle = (-1)^{|\phi_1|} \phi_1\psi_2 - \psi_1\phi_3 + h(\phi_1, \phi_2, \phi_3)$$

For any such  $\phi_1$ ,  $\phi_2$  and  $\phi_3$  we define the Massey product of the cohomology classes that they represent to be the cohomology class of the above cocycle. If  $\Delta$  is strictly coassociative then  $H$  and hence  $h(,,)$  may be chosen to be zero. This is the case in which Massey products are usually discussed, especially in topology, and is the case used in the previous two sections of this paper. From this case it is easy to see that varying the  $\phi_i$  by a coboundary or the  $\psi_i$  by a cocycle changes the Massey product by an element of the ideal of  $H^*(G)$  generated by  $\phi_1$  and  $\phi_3$ , so that the Massey product is well-defined modulo this ideal. Benson and Evens suggest another extreme case of interest [1], the case when  $G$  is finite and  $P_*$  is a minimal resolution. In this case the differential on the mod- $p$  cochains on  $P_*$  is trivial [5], so  $\psi_1$  and  $\psi_2$  may be chosen to be zero, and the Massey product is determined by  $h$ . We shall use this case below to determine Massey products in the mod- $p$  cohomology of finite abelian groups. First we discuss cyclic groups.

For  $G$  a cyclic group of order  $n$  with generator  $g$ , Cartan-Eilenberg describe a projective  $\mathbf{F}_p G$  resolution  $P_*$  for  $\mathbf{F}_p$  [3] in which  $P_i$  is free on one generator  $e_i$  with boundary map

$$d(e_i) = \begin{cases} (g-1)e_{i-1} & \text{for } i \text{ odd} \\ \sum_{j=0}^{n-1} g^j e_{i-1} & \text{for } i \text{ even.} \end{cases}$$

Cartan-Eilenberg also give a diagonal approximation for this resolution whose composite with the projection from  $P_* \otimes P_*$  to  $P_a \otimes P_b$  is given by

$$\Delta(e_{a+b}) = \begin{cases} e_a \otimes e_b & \text{for } a \text{ even,} \\ e_a \otimes ge_b & \text{for } a \text{ odd, } b \text{ even,} \\ \sum_{0 \leq i < j < n} g^i e_a \otimes g^j e_b & \text{for } a \text{ and } b \text{ odd.} \end{cases}$$

If  $n$  is a power of  $p$  then this resolution is minimal. We now describe a map  $H$  as above for this resolution.

**Proposition 5.** Let  $P_*$  with diagonal approximation  $\Delta$  be the Cartan-Eilenberg resolution for a cyclic group  $G$  of order  $n$  generated by  $g$  as above. Define a map  $H$  of degree one from  $P_*$  to  $(P_* \otimes P_* \otimes P_*)_*$  by defining its composite with projection to  $P_a \otimes P_b \otimes P_c$  to be

$$H(e_{a+b+c-1}) = \begin{cases} \sum_{0 \leq i < j < k < n} g^i e_a \otimes g^j e_b \otimes g^k e_c & \text{if } a, b \text{ and } c \text{ are all odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Then  $H$  satisfies the formula  $dH + Hd = (\Delta \otimes 1)\Delta - (1 \otimes \Delta)\Delta$ .

**Proof.** Check that the maps on each side of the above equation when composed with projection to  $P_a \otimes P_b \otimes P_c$  give the following map.

$$e_{a+b+c} \mapsto \begin{cases} \sum_{0 \leq i < j < n} (g^i e_a \otimes g^j e_b \otimes e_c - e_a \otimes g^{i+1} e_b \otimes g^{j+1} e_c) & \text{for } a, b \text{ and } c \text{ all odd,} \\ \sum_{0 \leq i < j < n} g^i e_a \otimes g^j e_b \otimes (1 - g^{j+1}) e_c & \text{for } a \text{ and } b \text{ odd,} \\ \sum_{0 \leq i < j < n} g^i e_a \otimes (g^{i+1} - g^j) e_b \otimes g^j e_c & \text{for } a \text{ and } c \text{ odd,} \\ \sum_{0 \leq i < j < n} (g^i - 1) e_a \otimes g^i e_b \otimes g^j e_c & \text{for } b \text{ and } c \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

*QED*

**Remark.** If the group  $G$  has order two then  $\Delta$  is strictly coassociative, and the empty sum occurring in the definition of  $H$  may be regarded as zero.

**Corollary 6.** Let  $G$  be a cyclic group of order  $p^m$ , where  $p^m > 2$ , and let  $u$  and  $t$  be generators for  $H^*(G)$  as in the introduction, where if  $m = 1$  then  $t$  is the Bockstein of  $u$ . The only Massey products that are defined in  $H^*(G)$  are  $\langle t^i u, t^j u, t^k u \rangle$ , which are defined modulo zero and take the following value.

$$\langle t^i u, t^j u, t^k u \rangle = \begin{cases} t^{i+j+k+1} & \text{for } p^m = 3, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** If  $p^m$  is greater than 3 then  $\binom{p^m}{3}$  is divisible by  $p$ , and the result follows easily from Proposition 5. If  $p^m = 3$  the argument requires slightly more care (and an explicit definition of the Bockstein map). Since this Corollary is well known we omit the remainder of the proof. *QED*

**Proposition 7.** Let  $G$  be a finite group expressible as  $G = G' \times G''$ , let  $P'_*$ ,  $P''_*$  be minimal resolutions for  $G'$  and  $G''$ , and let  $h'$ ,  $h''$  be functions as used above to define Massey products in  $H^*(G')$  and  $H^*(G'')$  respectively. Then  $(P'_* \otimes P''_*)_*$  is a minimal resolution for  $G$ , and the function  $h$  defined (for cochains  $\phi_i$  from  $P'_*$  to  $\mathbf{F}_p$  and  $\theta_i$  from  $P''_*$  to  $\mathbf{F}_p$ ) by

$$\begin{aligned} h(\phi_1 \otimes \theta_1, \phi_2 \otimes \theta_2, \phi_3 \otimes \theta_3) = & (-1)^{|\phi_2||\theta_1| + |\phi_3||\theta_1| + |\phi_3||\theta_2|} (h'(\phi_1, \phi_2, \phi_3) \\ & \otimes (\theta_1 \theta_2) \theta_3 + (-1)^{|\phi_1| + |\phi_2| + |\phi_3|} \phi_1(\phi_2 \phi_3) \otimes h''(\theta_1, \theta_2, \theta_3)) \end{aligned}$$

may be used to define Massey products in  $H^*(G)$ .

**Proof.** Let  $\Delta'$  be a diagonal approximation on  $P'_*$  and  $H'$  a chain homotopy between  $(\Delta' \otimes 1)\Delta'$  and  $(1 \otimes \Delta')\Delta'$  inducing  $h'$ , and define  $\Delta''$  and  $H''$  similarly. Now the map

which is defined to be  $H' \otimes (\Delta'' \otimes 1)\Delta'' + (-1)^i(1 \otimes \Delta')\Delta' \otimes H''$  on  $P'_i \otimes P''_j$  is a chain homotopy between  $(\Delta' \otimes 1 \otimes \Delta'' \otimes 1)(\Delta' \otimes \Delta'')$  and  $(1 \otimes \Delta' \otimes 1 \otimes \Delta'')(\Delta' \otimes \Delta'')$ . The composite of this map with the obvious map from  $P_*'^{\otimes 3} \otimes P_*''^{\otimes 3}$  to  $(P_*' \otimes P_*'')^{\otimes 3}$  which interlaces the factors is therefore a chain homotopy between  $(\Delta \otimes 1)\Delta$  and  $(1 \otimes \Delta)\Delta$  (where  $\Delta = (1 \otimes \tau \otimes 1)(\Delta' \otimes \Delta'')$ ), and this is the map which induces  $h$ . *QED*

**Remark.** Using Propositions 5 and 7 and Corollary 6 it is easy to calculate any Massey product in the mod- $p$  cohomology of a finite abelian group. We do not include the general formula because it is long and unilluminating. Corollary 8 and Theorem 9 contain examples of such calculations.

**Corollary 8.** *Let  $G$  be a finite abelian group. Then all the Massey products that are defined in  $H^*(G)$  contain zero unless  $p = 3$  and  $G$  has a direct summand of order three.*

### Examples.

In this short section we consider the 3-groups expressible as an extension with kernel cyclic of order three and quotient abelian of rank two. For these groups we determine all of the differentials in the Lyndon-Hochschild-Serre spectral sequence with  $\mathbf{F}_p$  coefficients, and hence the Poincaré series of their cohomology rings. The best examples among these (from the point of view of using our description of  $d_4$ ) are the non-metacyclic examples, but we shall include the others for completeness. The spectral sequence for the nonabelian group of order 27 and exponent 3, which we shall temporarily refer to as  $E$ , was first calculated by Huynh-Mui [6]. Our description of  $d_4$  greatly simplifies this calculation. The cohomology rings of the metacyclic examples are contained in work of Diethelm [4], who used a different spectral sequence. The other nontrivial cases of Theorem 9 are new.

Let  $H^*(C_{3^m} \oplus C_{3^n})$  be generated by  $y_1, y_2, x_1$  and  $x_2$  of degrees one, one, two and two respectively, where  $y_i$  and  $x_i$  are in the image of the inflation from projection onto the  $i$ th factor. The Bockstein maps  $y_1$  to  $x_1$  (resp. to zero) if  $m = 1$  (resp.  $m > 1$ ), and similarly for  $y_2$ . Modulo choice of generators, we need only consider four distinct extension classes in  $H^2(C_{3^m} \oplus C_{3^n})$ , 0,  $x_1$ ,  $x_1 + y_1y_2$ , and  $y_1y_2$ . The first two of these give abelian groups and the third gives a metacyclic group.

**Theorem 9.** *Let  $G$  be an extension with kernel cyclic of order three and quotient  $C_{3^m} \oplus C_{3^n}$ , and let  $\xi$  be its extension class. Then the Poincaré series for  $H^*(G)$  is one of the following power series.*

$$\begin{array}{ll} a) 1/(1-s)^3, & b) 1/(1-s)^2, \\ c) (1+s)/(1-s)(1-s^6), & d) (1+s+s^2)/(1-s^2)^2(1-s), \\ e) (1+s^2)/(1-s^6)(1-s)^2, & f) (1+s+2s^2+2s^3+s^4+s^5)/(1-s^6)(1-s). \end{array}$$

If  $\xi = 0$  then case a) occurs. If  $\xi = x_1$  then case b) occurs. If  $\xi = x_1 + y_1y_2$  then case b) occurs if  $n$  is greater than one and case c) occurs if  $n = 1$ . If  $\xi = y_1y_2$  then d) occurs if both  $m$  and  $n$  are greater than one, e) if one of  $m$  and  $n$  is one, and f) if  $m$  and  $n$  are both one.

**Proof.** As stated above, we examine the Lyndon-Hochschild-Serre spectral sequences for these extensions. It is convenient to introduce the bigraded Poincaré series for the pages

of the spectral sequence, defined by the following formula.

$$P_r(s, s') = \sum_{i,j} s^i s'^j \dim E_r^{i,j}$$

The Poincaré series that we wish to determine is equal to  $P_\infty(s, s)$ . For any central extension of  $p$ -groups the image of  $d_r$  is a finitely generated graded module for the kernel of  $d_r$  which in turn is a finitely generated graded  $\mathbf{F}_p$ -algebra. It follows that  $P_r$  and  $P_{r+1}$  satisfy a relation of the following form,

$$P_r(s, s') - P_{r+1}(s, s') = (s^r + s'^{r-1})f(s, s')/g(s, s'),$$

where  $f(s, s')$  is an integer polynomial and  $g(s, s')$  is a product of terms of the forms  $1 - s^k$  and  $1 - s'^l$ . This relation provides a useful check when calculating  $P_r(s, s')$ .

The  $E_2$ -page of the spectral sequence we are considering is  $\Lambda[u, y_1, y_2] \otimes \mathbf{F}_p[t, x_1, x_2]$ , and  $P_2 = 1/(1-s)^2(1-s')$ . All of the groups have an abelian subgroup of index three and an element in the cohomology of this subgroup that maps to  $t$  in the cohomology of the kernel. A simple Evens' norm map argument [5] now implies that in the spectral sequence  $t^3$  is a permanent cycle, and so  $E_7 = E_\infty$ . The first two differentials are given by  $d_2(u) = \xi$ ,  $d_2(t) = 0$ ,  $d_3(t) = \beta(\xi)$ .

If  $\xi$  is zero then the spectral sequence collapses. If  $\xi$  is either  $x_1$  or  $x_1 + y_1y_2$  then  $d_2$  is injective from odd rows to even rows, and  $P_3 = (1+s)/(1-s'^2)(1-s)$ . Now  $d_3(t)$  is zero if either  $\xi$  is  $x_1$  or  $\xi$  is  $x_1 + y_1y_2$  and  $m, n$  are both greater than one. If  $m = 1$  and  $n > 1$  then  $\beta(x_1 + y_1y_2) = (x_1 + y_1y_2)y_2$ , so that  $d_3(t)$  is zero in this case too. In the other cases when  $\xi = x_1 + y_1y_2$ ,  $d_3(t)$  is equal to  $-x_2y_1$  modulo the image of  $d_2$ , and in these cases  $P_4 = (1+(s-s^3)(1+s'^2)+ss'^4)/(1-s'^6)(1-s)$ . The  $E_4$ -page is concentrated in even vertical degree, so  $d_4$  must be zero. The only possible non-zero differential now is  $d_5(t^2y_1)$ , but  $d_5(t^2x_2y_1)$  is zero by Kudo's transgression theorem [8], and hence  $d_5(t^2y_1)$  is zero since multiplication by  $x_2$  is injective from  $E_5^{6,0}$  to  $E_5^{8,0}$ . Hence  $E_4 = E_\infty$ .

We now move on to the cases when  $\xi = y_1y_2$ . Here  $P_3$  is equal to  $(1+2s+s'(2s+s^2))/(1-s^2)(1-s^2)^2$ . If  $m$  and  $n$  are both greater than one, then  $d_3(t)$  is zero and so  $E_3 = E_\infty$ . If exactly one of  $m$  and  $n$  is equal to one, we may assume that  $n = 1$ . Now  $d_3(t) = -x_2y_1$ , and

$$\begin{aligned} P_4 = & (1+2s-s^3+s'(2s+s^2-s^4)+s'^2(2s-s^3) \\ & +s'^3(s+s^2-s^4)+2s'^4s+s'^5(s+s^2))/(1-s'^6)(1-s^2)^2. \end{aligned}$$

We now need to determine the images of  $tuy_1$ ,  $t^2y_1$ ,  $t^2y_2$  and  $t^2uy_1$  under  $d_4$ . From Proposition 7 and Corollary 6 it follows that

$$\langle x_2y_1, y_1, y_1y_2 \rangle = \langle y_1, y_1, y_1 \rangle x_2y_2 = 0,$$

and so by Theorem 3,  $d_4(tuy_1)$  is zero. Also  $(x_2y_1)y_1 = 0$ , whereas  $(-x_2y_1)y_2 = -(y_1y_2)x_2$ , and so Theorem 3 implies that  $d_4(t^2y_1) = 0$  and  $d_4(t^2y_2) = -ux_2^2y_1$ . The Poincaré series for the  $E_5$ -page is given by the following formula.

$$\begin{aligned} P_5 = & (1+2s-s^3+s'(2s+s^2-s^4-s^5)+s'^2(2s-s^3) \\ & +s'^3(s+s^2-s^4)+s'^4s+s'^5(s+s^2))/(1-s'^6)(1-s^2)^2. \end{aligned}$$

The remaining possibly non-zero differentials are  $d_5(t^2y_1)$  and  $d_5(t^2uy_1)$ , which we claim are zero, and then  $d_6(t^2uy_1)$ , which we also claim to be zero. It may be shown that  $d_5(t^2y_1) = 0$  either using a Kudo transgression theorem argument or by noting that no polynomial in  $x$  and  $x'$  can be hit, since  $E_5^{*,j} = E_6^{*,j}$  for  $j$  equal to 2 and 3, and each of these rows contains a free submodule for the subring of  $E_5^{*,0}$  generated by  $x$  and  $x'$ . Showing that  $d_5(t^2uy_1)$  is zero is more difficult. Since  $E_5^{6,1}$  is generated by  $uy_1y_2x_2^2$ , it suffices to show that a cohomology element may be chosen which yields  $uy_1y_2x_2^2$  and whose Bockstein yields a non-zero element in  $E_\infty^{7,1}$  (which is clearly isomorphic to  $E_5^{7,1}$ ). Let  $\hat{E}_r^{i,j}$  stand for the corresponding spectral sequence with integer coefficients. There is a Bockstein map from  $\hat{E}_r^{i,j}$  to  $\hat{E}_r^{i,j+1}$  if  $j > 0$  (resp.  $\hat{E}_r^{i+1,j}$  if  $j = 0$ ) converging to the usual Bockstein from mod- $p$  cohomology to integral cohomology. Under this Bockstein  $uy_1y_2$  maps to a generator for  $\hat{E}_\infty^{2,2}$  (one need only compute  $d_3$  to check this), which is clearly not divisible by three as a cohomology element because  $\hat{E}_\infty^{1,3}$  and  $\hat{E}_\infty^{0,4}$  are trivial. It follows that the Bockstein of an element yielding  $uy_1y_2$  must yield a nonzero multiple of  $ux_2y_1$  in  $E_*$ , and hence that  $d_5(t^2uy_1)$  cannot be  $uy_1y_2x_1^2$ , so must be zero. Similarly,  $d_6(t^2uy_1)$  must be in the kernel of the Bockstein from  $E_6^{7,0}$  to  $E_6^{8,0}$ , so can only be a multiple of  $x_1^3y_1$ . Comparison with the spectral sequence for the subextension with quotient  $C_{3^m} \oplus \{0\}$  shows now that  $d_6(t^2uy_1) = 0$ .

Now we consider the case when  $\xi = y_1y_2$  and  $m = n = 1$ . Here  $d_3(t)$  is equal to  $x_1y_2 - x_2y_1$ , and  $P_4$  is given by the following formula.

$$\begin{aligned} P_4 = & (1 + 2s - s^3 + s'(2s + s^2 - 2s^4 + s^6) + s'^2(2s - s^3) \\ & + s'^3(s^2 + s^3 - 2s^4 + s^6) + 2s'^4s + s'^5(s^2 + s^3))/(1 - s^2)^2(1 - s'^6) \end{aligned}$$

We need to determine the images under  $d_4$  of  $tu(x_1y_2 - x_2y_1)$ ,  $t^2y_1$ ,  $t^2y_2$  and  $t^2u(x_1y_2 - x_2y_1)$ . From Proposition 7 and Corollary 6 we see that

$\langle x_1y_2 - y_1x_2, x_1y_2 - y_1x_2, y_1y_2 \rangle = \langle y_1, y_1, y_1 \rangle x_2^2y_2 - x_1^2y_1 \langle y_2, y_2, y_2 \rangle = x_1x_2^2y_2 - x_1^2x_2y_1$ , and hence that  $d_4(t^i u(x_1y_2 - x_2y_1)) = it^{i-1}(x_1x_2^2y_2 - x_1^2x_2y_1)$ . Note that modulo multiples of  $x_1y_2 - x_2y_1$ ,  $x_1x_2^2y_2 - x_1^2x_2y_1$  is equal to the first reduced power of  $x_1y_2 - x_2y_1$ , which we know must be hit by some differential. Also note that

$$(x_1y_2 - x_2y_1)y_1 = -(y_1y_2)x_1 \quad (x_1y_2 - x_2y_1)y_2 = -(y_1y_2)x_2,$$

from which it follows that  $d_4(t^2y_i) = u(x_1y_2 - x_2y_1)x_i$ . The Poincaré series for  $E_5$  page is the following expression.

$$\begin{aligned} P_5 = & (1 + 2s - s^3 - s^7 + s'(2s + s^2 - 2s^4 - 2s^5 + s^6) + s'^2(2s - s^3 - s^7) \\ & + s'^3(s^2 - 2s^4 + s^6) + s'^4s + s'^5s^2)/(1 - s^2)^2(1 - s'^6) \end{aligned}$$

It remains now to calculate  $d_5(t^2(x_1y_2 - x_2y_1))$ , which is  $x_1^3x_2 - x_2^3x_1$  by Kudo's transgression theorem, and  $d_5(t^2uy_1y_2)$ , which must be a non-zero multiple of  $u(x_1^3y_2 - x_2^3y_1)$  to ensure that  $E_6^{*,1}$  has a well defined module structure for  $E_6^{*,0}$ . The Poincaré series for  $E_6$ , which is equal to  $E_\infty$ , is the following.

$$\begin{aligned} P_6 = & (1 + 2s + s^2 + s^3 + s^4 + s^5 + s^6 + s'(2s + s^2 + 2s^3 - s^4) \\ & + s'^2(2s + s^3 + s^5) + s'^3(s^2 - s^4))/(1 - s^2)(1 - s'^6) \end{aligned}$$

*QED*

## References.

- [1] D. J. Benson, *Representations and Cohomology I*, Cambridge Studies in Advanced Mathematics, Cambridge University Press (1991).
- [2] H. Cartan, Sur les groupes d'Eilenberg-MacLane  $H(\pi, n)$  I and II, *Proc. Nat. Acad. Sci. Amer.* **40** (1954), 467–471 and 704–707.
- [3] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press (1956).
- [4] T. Diethelm, The mod- $p$  cohomology rings of the non abelian split metacyclic  $p$ -groups, *Arch. Math. Basel* **44** (1985) 29–38.
- [5] L. Evens, *The Cohomology of Groups*, Oxford Mathematical Monographs, Oxford University Press (1991).
- [6] Huỳnh Mùi, The mod- $p$  cohomology algebra of the extra-special group  $E(p^3)$ , preprint (1983).
- [7] D. M. Kan and W. P. Thurston, Every connected space has the homology of a  $K(\pi, 1)$ , *Topology* **15** (1976) 253–258.
- [8] T. Kudo, A transgression theorem, *Mem. Fac. Sci. Kyusyu Univ. Series A*, **9** (1956) 79–81.
- [9] J. P. Serre, Cohomologie modulo 2 des complexes d'Eilenberg-MacLane, *Comm. Math. Helv.* **27** (1953) 192–232.
- [10] N. E. Steenrod, Products of cocycles and extensions of mappings, *Ann. Math.* **48** (1947) 290–320.